ON THE UNITARY DUAL OF $Spin(2n, \mathbb{C})$

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ABSTRACT. In this paper we begin a systematic study of the unitarity question for genuine representations of the group $Spin(2n, \mathbb{C})$. The main result is that we find a class of unitary representations (mostly isolated) analogous to the special unipotent representations defined by D. Barbasch and D. Vogan. In particular the full unitary dual of $Spin(2n, \mathbb{C})$ should be obtainable from this set by complementary series.

INTRODUCTION

The unitary dual of a Lie group plays an important role in harmonic analysis and automorphic forms. Motivated by conjectures of Arthur, a class of representations called special unipotent is defined in [BV1] for any complex reductive Lie group. Typically these are representations of the adjoint group. In [B] these representations were shown to be unitary in the case of complex classical groups. The most basic cases for the classical groups are spherical irreducible, and so they have maximal primitive ideal. This makes it possible to write character formulas. However these results do not cover the case of genuine representations of the connected simply connected versions of these groups, $Spin(2n, \mathbb{C})$ and $Spin(2n + 1, \mathbb{C})$.

In this paper we lay the groundwork for determining the unitary dual for the complex Spin groups. The main result is that we find a class of unitary representations (mostly isolated) analogous to the special unipotent representations in [BV1]. In particular the full unitary dual should be obtainable from this set by complementary series.

In this paper we begin a systematic study of the unitarity question for genuine representations of the Spin groups; we consider the case of $Spin(2n, \mathbb{C})$. First we find necessary conditions for unitarity (section 2) using the technique of bottom layer K-types (see Definition 4.12 of [SV]). Proposition 2.2 reduces considerations to the case when the representation has *spin* as lowest K-type. Because these representations do not have maximal primitive ideal, we do not have enough control over the K-types to get necessary conditions for unitarity that are also sufficient. The precise results are contained in Theorem 2.1.

We then consider the case when the lowest K-type is the *spin* representation. The Kazhdan-Lusztig conjectures for non-integral infinitesimal character allow us to reduce it to the case when the integral system determined by the infinitesimal

character is of type $D_n \times D_n$ (section 2, Lemma 2.1). To obtain further necessary conditions would require a difficult analysis of the hermitian form on certain K-types; we plan to pursue this in future work.

In section 3 we turn to the problem of finding sufficient conditions for unitarity. Following [B] we consider the special case when the length of the infinitesimal character is as small as possible subject to the condition that the wavefront set be the closure of a given (special) nilpotent orbit and has the *spin* representation as lowest K-type. These orbits are the same as those considered in [BV1] and in [B]; the precise statements are in Lemma 3.1, Proposition 3.1 and Proposition 3.2. We then show that these representations are unitary following the techniques of [B].

We expect that these are all the unitary representations with the *spin* representation as lowest K-type and this type of infinitesimal character. Furthermore the complementary series arising from such parameters should be fairly easy to determine. This is part of ongoing research.

1. General results and notation

Let G be a connected simply connected complex semisimple Lie group viewed as a real group and let $\mathfrak{g}_0$ denote its Lie algebra. Let K be a fixed maximal compact subgroup and let B be a Borel subgroup. Then $T = K \cap B$ is a maximal torus in K. If $\mathfrak{t}_0$ is its Lie algebra, then $\mathfrak{a}_0 = i\mathfrak{t}_0$ is a maximally split component. If $A = \exp \mathfrak{a}_0$, then $H = TA$ is a Cartan subgroup of G. Let $\mathfrak{h}_0$ denote its Lie algebra and let $\mathfrak{g}, \mathfrak{h}, \mathfrak{t}$ and $\mathfrak{a}$ denote the complexifications of $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{t}_0$ and $\mathfrak{a}_0$, respectively. Denote by Δ the roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$ and by Δ^+ the set of positive roots corresponding to the fixed Borel subgroup $B = HN^+$. Let W be the Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_0)$ and let $\Lambda = \{\mu \in \mathfrak{h}_0^* : (\mu, \check{\alpha}) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ be the set of weights of $\mathfrak{g}_0$. Recall that $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ for every α in Δ.

We now describe the classification of $(\mathfrak{g}, K)$-modules. Let $\lambda_1, \lambda_2 \in \mathfrak{h}_0^*$ be such that $\lambda_1 - \lambda_2 = \mu \in \Lambda$ and let $\nu = \lambda_1 + \lambda_2$. Define a character $\mathbb{C}_{(\lambda_1,\lambda_2)}$ of H as $\mathbb{C}_{(\lambda_1,\lambda_2)}|_T = \mathbb{C}_\mu$ and $\mathbb{C}_{(\lambda_1,\lambda_2)}|_A = \mathbb{C}_\nu$, and extend $\mathbb{C}_{(\lambda_1,\lambda_2)}$ to a character of B by making it trivial on N^+. Now define

$$X(\lambda_1, \lambda_2) = \left[\mathrm{Ind}_B^G \left(\mathbb{C}_{(\lambda_1,\lambda_2)} \right) \right]_{K-\mathrm{finite}},$$

and let $\bar{X}(\lambda_1, \lambda_2)$ denote the unique irreducible subquotient of $X(\lambda_1, \lambda_2)$ containing the K-type with extremal weight μ. Then the following theorem summarizes the classification of $(\mathfrak{g}, K)$-modules.

Theorem 1.1. *(Parthasarathy-Rao-Varadarajan, Zhelobenko). Fix (λ_1, λ_2) and (λ_1', λ_2') as before. The following statements are equivalent:*

  (i) *$X(\lambda_1, \lambda_2)$ and $X(\lambda_1', \lambda_2')$ have the same composition factors with multiplicities.*
 (ii) *$\bar{X}(\lambda_1, \lambda_2) \simeq \bar{X}(\lambda_1', \lambda_2')$.*
(iii) *There is $w \in W$ such that $w\lambda_1 = \lambda_1'$ and $w\lambda_2 = \lambda_2'$.*

Moreover, any irreducible $(\mathfrak{g}, K)$-module is isomorphic to an $\bar{X}(\lambda_1, \lambda_2)$.

Proof. See [D].   □

Recall that a hermitian form on a $(\mathfrak{g}, K)$-module (π, V) is called $\mathfrak{g}$-invariant if

$$(\pi(X)\, v, w) = -(v, \pi(X^*)\, w)$$

where $X = X_1 + jX_2$ and $X^* = X_1 - jX_2$ for $X_1, X_2 \in \mathfrak{g}_0$. Here, j denotes the action of $\sqrt{-1}$ coming from the complexification of $\mathfrak{g}_0$. It is a well-known result of

Harish-Chandra that the problem of classifying the unitary irreducible G-modules is equivalent to the problem of classifying the irreducible $(\mathfrak{g}, K)$-modules admitting a positive definite $\mathfrak{g}$-invariant form. The following well-known theorem characterizes the $(\mathfrak{g}, K)$-modules that admit a non-degenerate $\mathfrak{g}$-invariant hermitian form.

Theorem 1.2. *$\bar{X}(\lambda_1, \lambda_2)$ admits a non-degenerate $\mathfrak{g}$-invariant hermitian form if and only if there is $w \in W$ such that $w\mu = \mu$ and $w\nu = -\bar{\nu}$, where $\mu = \lambda_1 - \lambda_2$ and $\nu = \lambda_1 + \lambda_2$.*

Proof. See [D] or Theorem 2.4 of [B] for a proof. □

For simplicity we will refer to $\mathfrak{g}$-invariant non-degenerate hermitian forms as just non-degenerate forms. If $\bar{X}$ admits a non-degenerate form, we will say that $\bar{X}$ is hermitian. Also, in all cases, we will assume that the form is normalized to be positive on the lowest K-type. The following theorem allows us to reduce the classification of irreducible unitary $(\mathfrak{g}, K)$-modules to the case of real infinitesimal character.

Theorem 1.3. *Suppose $\bar{X}(\lambda_1, \lambda_2)$ is unitary. Then there is a parabolic subgroup P, a unitary representation $\bar{X}_R$ with real infinitesimal character and a unitary character χ such that*

$$\bar{X}(\lambda_1, \lambda_2) = \operatorname{Ind}_P^G \left[\bar{X}_R \otimes \chi \otimes 1 \right].$$

Proof. This is a special case of a more general result; see [V1] and its references. See also Corollary 2.5 of [B] for a self-contained proof in this case. □

In view of this theorem we may assume throughout the rest of the paper that the infinitesimal character of $\bar{X}(\mu, \nu)$ is real.

2. Necessary conditions for unitarity coming from bottom layer K-types

In this section we use the notion of bottom layer K-type to deduce some necessary conditions for unitarity. We begin by recalling the definition and some known facts about bottom layer K-types.

Assume μ is dominant for Δ^+. Let $P_\mu = M_\mu N_\mu$ be the real parabolic subgroup of G defined by

$$\Delta(\mathfrak{m}_\mu, \mathfrak{a}) = \{\alpha \in \Delta \: : \: (\alpha, \mu) = 0\},$$

$$\Delta(\mathfrak{n}_\mu, \mathfrak{a}) = \{\alpha \in \Delta \: : \: (\alpha, \mu) > 0\}.$$

Let $P = MN$ be a parabolic subgroup containing P_μ. Given a parameter (λ_1, λ_2) such that $\mu = \lambda_1 - \lambda_2$, we can define a standard module $X_M(\lambda_1, \lambda_2)$ for $(\mathfrak{m}, M \cap K)$ with lowest $M \cap K$-type subquotient $\bar{X}_M(\lambda_1, \lambda_2)$. Then, we have

$$X(\lambda_1, \lambda_2) = \operatorname{Ind}_P^G \left[X_M(\lambda_1, \lambda_2) \otimes 1 \right]$$

and $\bar{X}(\lambda_1, \lambda_2)$ is the lowest K-type subquotient of $\operatorname{Ind}_P^G \left[\bar{X}_M(\lambda_1, \lambda_2) \otimes 1 \right]$.

Definition 2.1 (see Definition 4.12 of [SV]). A K-type γ is called $\mathfrak{p}$-bottom layer for $X(\lambda_1, \lambda_2)$ if

$$\gamma = \mu + \sum_{\alpha \in \Delta(\mathfrak{m})} m_\alpha . \alpha \quad \text{for } m_\alpha \in \mathbb{Z} .$$

Now, suppose (π, V) is a $(\mathfrak{g}, K)$-module admitting an invariant hermitian form and let $\gamma \in \hat{K}$. Then, let $[\gamma : V]$ denote the multiplicity of γ in V and let $[\gamma : V]_\pm$ denote the dimension of the $\pm$-signature of the γ-isotypic component of V. With the notation as above assume that $\bar{X}_M(\lambda_1, \lambda_2)$ is hermitian. Then so is $\bar{X}(\lambda_1, \lambda_2)$ and we have

Proposition 2.1 (D. Vogan, [V2]). *Let γ be a $\mathfrak{p}$-bottom layer K-type. Then*

$\quad$ (i) $\quad [\gamma : X(\lambda_1, \lambda_2)] \quad = \quad [\gamma : X_M(\lambda_1, \lambda_2)],$

$\quad$ (ii) $\quad [\gamma : \bar{X}(\lambda_1, \lambda_2)]_\pm \quad = \quad [\gamma : \bar{X}_M(\lambda_1, \lambda_2)]_\pm.$

Proof. As for Theorem 1.3, this is a special case of a more general result for real reductive groups. See Proposition 2.7 of [B] for a proof of this special case. $\quad\square$

From now on we will assume that $G = Spin(2n, \mathbb{C})$, that is, G is a simply connected twofold covering of $SO(2n, \mathbb{C})$. As above, let $\mathfrak{g}_0$ denote the Lie algebra of G and let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. Then, $\mathfrak{h}_0 \simeq \mathbb{C}^n$ and $\Delta^+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\}$ where $\{\epsilon_1, \ldots, \epsilon_n\}$ is an orthonormal basis of $\mathfrak{h}_0^*$. The corresponding system of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ is given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. The Weyl group W acts on $\mathfrak{h}_0^*$ as the group of all permutations and sign changes on $\{\epsilon_1, \ldots, \epsilon_n\}$ but only an even number of sign changes are allowed. Also, it is easy to see that $\mu = \sum_{j=1}^{n} s_j \epsilon_j$ is a weight of $\mathfrak{g}_0$ if and only if the coordinates s_j are all integers or all half-integers. When all the coordinates of μ are integers the representation $\bar{X}(\mu, \nu)$ factors through the group $SO(2n, \mathbb{C})$. These representations are studied in [B]. Hence, since we only want to consider genuine representations of $Spin(2n, \mathbb{C})$, we will assume that all coordinates of μ are half-integers. Also, up to conjugation by elements in the Weyl group, we can assume that $\mu = \mu^\pm = \left(\frac{2r+1}{2}, \ldots, \frac{2r+1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\right)$. Now, since the $(\mathfrak{g}, K)$-modules $\bar{X}(\mu^+, \nu)$ and $\bar{X}(\mu^-, \nu)$ are related by the automorphism of G induced by the permutation of the simple roots α_{n-1} and α_n, we may restrict our attention to the case $\mu = \mu^+$. Then, from now on we will assume that $\mu = \left(\frac{2r+1}{2}, \ldots, \frac{2r+1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}\right)$.

Let n_j denote the number of times that the coordinate $(2j + 1)/2$ occurs in μ. This defines a partition $\pi = (n_r, n_{r-1}, ..., n_0)$ of n. Similarly, if $\nu \in \mathfrak{h}_0^*$, write $\nu = (\nu_r, \nu_{r-1}, ..., \nu_0)$ where ν_j is the restriction of ν to the jth block of π. If $P_\mu = M_\mu N_\mu$ is the parabolic subgroup of G defined above, we have $M_\mu = GL(n_r, \mathbb{C}) \times ... \times GL(n_0, \mathbb{C})$. Now consider the parabolic subgroup $P = MN$ of G, containing P_μ, with Levi factor $M = GL(n_r, \mathbb{C}) \times ... \times GL(n_1, \mathbb{C}) \times Spin(2n_0, \mathbb{C})$. Then, for each $j = 0, 1, ..., r$ let $\bar{X}_j = \bar{X}(\mu_j, \nu_j)$ be the irreducible representation of the jth factor of M defined by the parameters (μ_j, ν_j). In this setting we have the following result.

Proposition 2.2. $\quad$ (i) $\bar{X}(\mu, \nu)$ *is a subquotient of* $\mathrm{Ind}_P^G \left(\bar{X}_r \otimes \cdots \otimes \bar{X}_0\right).$

$\quad$ (ii) $\bar{X}(\mu, \nu)$ *is hermitian if and only if each factor* $\bar{X}(\mu_i, \nu_i)$ $(i = 0, 1, \ldots, r)$ *is hermitian.*

$\quad$ (iii) *If* $\bar{X}_i$ *is not unitary for some* $i \geq 1$, *then* $\bar{X}(\mu, \nu)$ *is not unitary.*

Proof. Parts (i) and (ii) are easy consequences of the results in section 1. We will only prove (iii). Let us assume that $\bar{X}(\mu, \nu)$ admits a $\mathfrak{g}$-invariant non-degenerate hermitian form. Then, in view of (ii), every representation $\bar{X}(\mu_i, \nu_i)$ $(i = 0, 1, \ldots, r)$ also admits a $\mathfrak{g}$-invariant non-degenerate hermitian form. Assume that all hermitian forms are normalized to be positive on the lowest K-type.

Suppose that for some $j \geq 1$ the representation $\bar{X}(\mu_j, \nu_j)$ is not unitary. Then, since $\bar{X}(\mu_j, \nu_j)$ is an irreducible almost spherical representation of $GL(n_j, \mathbb{C})$ of type μ_j, (see p. 453 of [V1] for the definition), it follows from Theorem 7.8 of [V1] that the hermitian form is negative on a K-type of the form $\mu_j + w_q$, where $w_q = (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$ (q ones and q negative ones). Set $\gamma = \mu_r \otimes \cdots \otimes \mu_{j+1} \otimes (\mu_j + w_q) \otimes \mu_{j-1} \otimes \cdots \otimes \mu_0$. Then γ is a $M \cap K$-type of $\bigotimes\limits_{j=0}^{r} \bar{X}(\mu_j, \nu_j)$ and, since $j \geq 1$, it is also dominant for K. Therefore γ is a bottom layer K-type for P. Then it follows from Proposition 2.1 that

$$\left[\gamma : \bar{X}(\mu, \nu) \right]_- = \left[\gamma : \operatorname{Ind}_P^G \left(\bigotimes_{j=0}^{r} \bar{X}(\mu_j, \nu_j) \right) \right]_- > 0.$$

Hence $\bar{X}(\mu, \nu)$ is not unitary, as we wanted to show. □

It follows from (iii) of Proposition 2.2 that if $\bar{X}(\mu, \nu)$ is unitary, the representations $\bar{X}(\mu_i, \nu_i)$ for $i = 1, 2, \ldots, r$ are all unitary almost spherical representations of $GL(n_i, \mathbb{C})$ of type μ_i. Hence, it follows from [V1] that $\bar{X}(\mu_i, \nu_i)$ (for $i = 1, 2, \ldots, r$) is induced from unitary characters or Stein complementary series.

This, together with Theorem 1.3, implies that we should study hermitian representations $\bar{X}_0$ of $Spin(2n_0, \mathbb{C})$ with $\mu_0 = \left(\frac{1}{2}, \ldots, \frac{1}{2} \right)$ and ν_0 real. By Theorem 1.2, since $\bar{X}(\mu_0, \nu_0)$ is hermitian, $-\nu_0$ is a permutation of ν_0. Write $\nu_0 = (a_1, \ldots, a_{n_0})$ and break ν_0 up into maximal subsets in such a way that if a_i and a_j belong to the same subset, then $a_i - a_j \in 2\mathbb{Z}$. Let ν^j $(j = 1, \ldots, s)$ denote the subsets in which ν_0 is partitioned and permute ν_0 to a new sequence $\nu_0' = (\nu^1, \nu^2, \ldots, \nu^s)$. If p_j denotes the cardinality of $\nu^j, \pi = (p_1, p_2, \ldots, p_s)$ gives a partition of n_0. For each $j = 1, 2, \ldots, s$, let μ^j denote the restriction of μ_0 to the jth block of π and set $\lambda_1^j = \frac{1}{2}(\nu^j + \mu^j)$ and $\lambda_2^j = \frac{1}{2}(\nu^j - \mu^j)$. For each $j = 1, \ldots, s$, we consider the set $\Delta(\lambda_1^j)$ of all roots of $Spin(2n_0, \mathbb{C})$ within the jth block of π that are integral for λ_1^j. Then, the following facts are easy to verify.

Lemma 2.1. *If $\nu^j = (a_1, \ldots, a_{p_j})$, we have*

(i) *$\Delta(\lambda_1^j)$ is of type D if and only if $a_l + a_k \in 2\mathbb{Z} + 1$ for all $1 \leq l, k \leq p_j$. In particular, $a_i \in \mathbb{Z} + \frac{1}{2}$ for all $1 \leq i \leq p_j$.*

(ii) *If $\Delta(\lambda_1^j)$ is of type D, $-\nu^j$ is (up to a permutation) one of the other ν^j.*

(iii) *There exists at most one j for which ν^j (and therefore $-\nu^j$) is of type D. For all the other j's, $\Delta(\lambda_1^j)$ is of type A.*

Without loss of generality we can assume that $\Delta(\lambda_1^1)$ is of type D and that $-\nu^1$ is ν^2. Then set $p = p_1 = p_2$, $\tilde{\nu} = (\nu^1, -\nu^1)$, $\tilde{\mu} = (\mu^1, \mu^2)$ and, let $\bar{X}(\tilde{\mu}, \tilde{\nu})$ be the irreducible representation of $Spin(2p, \mathbb{C})$ defined by $(\tilde{\mu}, \tilde{\nu})$. Similary, for $j = 3, \ldots, s$, let $\bar{X}_j = \bar{X}(\mu^j, \nu^j)$ be the irreducible almost spherical representation of $GL(p_j, \mathbb{C})$ defined by (μ^j, ν^j).

Set $G_0 = Spin(2n_0, \mathbb{C})$ and let $P_0 = M_0 N_0$ be the parabolic subgroup of G_0 with $M_0 = Spin(2p, \mathbb{C}) \times \cdots \times GL(p_s, \mathbb{C})$. It follows from the Kazhdan-Lusztig conjecture for non-integral infinitesimal character that

(1) $$\bar{X}(\mu_0, \nu_0) = \operatorname{Ind}_{P_0}^{G_0} \left(\bar{X}(\tilde{\mu}, \tilde{\nu}) \otimes \bar{X}_3 \otimes \cdots \otimes \bar{X}_s \right).$$

We will now establish some necessary conditions for the unitarity of our original $(\mathfrak{g}, K)$-module $\bar{X}(\mu, \nu)$. These conditions will be expressed in terms of the parameter $(\tilde{\mu}, \tilde{\nu})$ defined above. Recall that $\tilde{\nu} = (\nu^1, -\nu^1)$ where $\nu^1 = (a_1, \dots, a_p)$ is such that $a_i \in \mathbb{Z} + \frac{1}{2}$ and $a_i - a_j \in 2\mathbb{Z}$ for all $1 \leq i, j \leq p$. Interchanging the roles of ν^1 and $-\nu^1$ if necessary, we can assume that

$$(2) \qquad \nu^1 = (a, ..., a, a-2, ..., a-2, ..., -b+2, ..., -b+2, -b, ..., -b)$$

where $a = (4k+1)/2$ and $b = (4l-1)/2$, with k and l non-negative integers. Let r_j be the number of times that the coordinate $(4j+1)/2$ occurs in ν^1 (here $j = 0, 1, \dots, k$), and for $j = 1, 2, \dots, l$, let s_j denote the number of times that the coordinate $-(4j-1)/2$ occurs in ν^1. Then, we have the following result.

Theorem 2.1. $\bar{X}(\mu, \nu)$ *is unitary only if* $r_0 \neq 0$ *and the inequalities* $r_0 \geq r_1 \geq \cdots \geq r_k$ *and* $r_0 \geq s_1 \geq \cdots \geq s_l$ *hold.*

Proof. Set $G_1 = Spin(2p, \mathbb{C})$ and let $\bar{X}(\tilde{\mu}, \tilde{\nu})$ be the irreducible representation of G_1 defined above. Recall that $\tilde{\mu} = (\frac{1}{2}, \dots, \frac{1}{2})$ and $\tilde{\nu} = (\nu^1, -\nu^1)$ where ν^1 is as in (2). Let $P_1 = M_1 N_1$ be the parabolic subgroup of G_1 defined by $\tilde{\mu}$, that is, $M_1 = GL(2p, \mathbb{C})$ is the centralizer of $\tilde{\mu}$. Given the parameter $(\tilde{\mu}, \tilde{\nu})$, we can define a standard module $X_{M_1}(\tilde{\mu}, \tilde{\nu})$ for $(\mathfrak{m}_1, M_1 \bigcap K_1)$ with lowest $M_1 \bigcap K_1$-type subquotient $\bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu})$. Then, $\bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu})$ is an irreducible almost spherical representation of $GL(2p, \mathbb{C})$ and $\bar{X}(\tilde{\mu}, \tilde{\nu})$ is a subquotient of

$$(3) \qquad \qquad \mathrm{Ind}_{P_1}^{G_1} \left(\bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu}) \otimes 1 \right).$$

It follows from Section 6 of [V1] that there exists a partition $\pi = (q_1, q_1, ..., q_m, q_m)$ of $2p$ and irreducible finite dimensional almost spherical representations V_j and V_j^* of $GL(q_j, \mathbb{C})$ $(j = 1, 2, ..., m)$ such that

$$(4) \qquad \qquad \bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu}) = \mathrm{Ind}_{P(\pi)}^{M_1} \left[(V_1 \otimes V_1^*) \otimes ... \otimes (V_m \otimes V_m^*) \right]$$

where $P(\pi)$ is the parabolic subgroup of $M_1 = GL(2p, \mathbb{C})$ associated to the partition π. The representations V_j and V_j^* are obtained as follows. Extract from ν^1 one coordinate from each entry, this gives a decreasing sequence $\nu_1 = (a, a-2, \dots, \frac{1}{2}, -\frac{3}{2}, \dots, -b+2, -b)$ of length q_1. Next, extract one coordinate from each entry in the remainder. This gives a decreasing sequence ν_2 of length q_2. Continue this process until all the coordinates of ν^1 have been extracted. This gives m decreasing sequences $\nu_1, \nu_2, \dots, \nu_m$. Now, for each $j = 1, 2, ..., m$, set $\mu_j = (\frac{1}{2}, ..., \frac{1}{2})$ and let V_j (respectively V_j^*) denote the irreducible almost spherical representation of $GL(q_j, \mathbb{C})$ defined by the parameter (μ_j, ν_j) (respectively $(\mu_j, -\nu_j)$). Then, since the difference between any two consecutive coordinates of ν_j is in $2\mathbb{N} - \{0\}$, it follows from Lemma 11.11 of [V1] that V_j and V_j^* are finite dimensional. We recall that $\dim V_j = 1$ if and only if the difference between any two consecutive coordinates of ν_j is exactly 2 (see Lemma 11.11 of [V1]).

The condition in Theorem 2.1 is that all V_j have dimension one and, if we write ν_j as $(i+2t, i-2+2t, ..., -(i-2)+2t, -i+2t)$ with $t \geq 0$, then $2t \leq i+1$. This is precisely the condition that the hermitian form be positive on the bottom layer K-type $(\frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$.

We will first show that if any V_j has dimension greater than one, then $\bar{X}(\mu, \nu)$ is not unitary. Without loss of generality we may assume that $\dim V_1 > 1$. Let $Q = LN$ denote the parabolic subgroup of $M_1 = GL(2p, \mathbb{C})$ with Levi factor $L = L_1 \times (GL(q_2, \mathbb{C}) \times GL(q_2, \mathbb{C})) \times ... \times (GL(q_m, \mathbb{C}) \times GL(q_m, \mathbb{C}))$ where $L_1 = GL(2q_1, \mathbb{C})$

and let Q_1 denote the parabolic subgroup of L_1 with Levi factor $GL(q_1, \mathbb{C}) \times GL(q_1, \mathbb{C})$. Then, in view of (4) and induction in stages, we have

$$(5) \qquad \bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu}) = \mathrm{Ind}_Q^{M_1} \left[\left(\mathrm{Ind}_{Q_1}^{L_1}(V_1 \otimes V_1^*) \right) \otimes \left(\bigotimes_{j=2}^m (V_j \otimes V_j^*) \right) \right].$$

Now, since dim $V_1 > 1$, it follows from the proof of Theorem 7.8 of [V1] that the hermitian form on $\mathrm{Ind}_{Q_1}^{L_1}(V_1 \otimes V_1^*)$ is negative on the K-type $\gamma_{2q_1} = \left(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right)$ (here K is the unitary group $U(2q_1)$). Hence, the hermitian form on $\bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu})$ is negative on the K-type $\gamma_{2p} = \left(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right)$ (here $K = U(2p)$). Now, γ_{2p} is dominant for K_1, the maximal compact subgroup of G_1, and since $\gamma_{2p} = \tilde{\mu} + w_1$, where $w_1 = (1, 0, \ldots, 0, -1)$ is a root of M_1, γ_{2p} is a bottom layer K-type for $P_1 = M_1 N_1$. Hence, we have

$$(6) \qquad \left[\gamma_{2p} : \bar{X}(\tilde{\mu}, \tilde{\nu}) \right]_- = \left[\gamma_{2p} : \mathrm{Ind}_{P_1}^{G_1} \left(\bar{X}_{M_1}(\tilde{\mu}, \tilde{\nu}) \otimes 1 \right) \right]_- > 0.$$

Now set $\gamma_{2n_0} = \left(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right) \in \hat{K}_0$ where K_0 is the maximal compact subgroup of $G_0 = Spin(2n_0, \mathbb{C})$. Then, (1) and (6) imply that $\left[\gamma_{2n_0} : \bar{X}(\mu_0, \nu_0) \right]_- > 0$. Finally, since

$$\gamma_{2n} = \mu_r \otimes \cdots \otimes \mu_1 \otimes \gamma_{2n_0} = \left(\frac{2r+1}{2}, \ldots, \frac{2r+1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right)$$

is an $M \bigcap K$-type of $\bar{X}_r \otimes \cdots \otimes \bar{X}_1 \otimes \bar{X}(\mu_0, \nu_0)$ and it is dominant for K, it is a bottom layer K-type for $P = MN$, where P is as in Proposition 2.2. Then, it follows that

$$\left[\gamma_{2n} : \bar{X}(\mu, \nu) \right]_- = \left[\gamma_{2n} : \mathrm{Ind}_P^G \left(\bar{X}_r \otimes \cdots \otimes \bar{X}_1 \otimes \bar{X}(\mu_0, \nu_0) \right) \right]_- > 0.$$

Therefore, $\bar{X}(\mu, \nu)$ is not unitary as we claimed.

We are now in a position to prove Theorem 2.1. We begin by showing that if $r_0 = 0$, then $\bar{X}(\mu, \nu)$ is not unitary. Suppose that $r_0 = 0$. Define $R(\nu^1) = \{ r_j : r_j \neq 0, \ 1 \leq j \leq k \}$ and $S(\nu^1) = \{ s_i : s_i \neq 0, \ 1 \leq i \leq l \}$. If both $R(\nu^1)$ and $S(\nu^1)$ are not empty we have dim $V_1 > 1$; hence by the above argument, $\bar{X}(\mu, \nu)$ is not unitary. On the other hand, if $R(\nu^1) \neq \emptyset$ and $S(\nu^1) = \emptyset$ and, furthermore, dim $V_1 = 1$, the parameter ν_1 is of the form

$$(7) \qquad \begin{aligned} \nu_1 &= \left(2m + \frac{1}{2}, 2(m-1) + \frac{1}{2}, \ldots, 2(m-j) + \frac{1}{2} \right) \\ &= (j + 2t, j - 2 + 2t, \ldots, -(j-2) + 2t, -j + 2t) \end{aligned}$$

for some integers m and j such that $1 \leq m \leq k$ and $0 \leq j \leq m - 1$ and $t = (2(2m - j) + 1)/4$. Then, since $m - j \geq 1$ we have $2t > j + 1$ (observe that the point $(j+1)/2$ is where the trivial representation occurs). Therefore, it follows from Lemma 12.12 of [V1] that the hermitian form on $\mathrm{Ind}_{Q_1}^{L_1}(V_1 \otimes V_1^*)$, which is positive on the lowest K-type, is negative on the K-type $\gamma_{2q_1} = \left(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right)$. Hence, by arguing as above, we can show from this fact that $\bar{X}(\mu, \nu)$ is not unitary. The remaining case, $R(\nu^1) = \emptyset$ and $S(\nu^1) \neq \emptyset$, can be reduced to the previous case by interchanging the roles of ν_1 and $-\nu_1$.

For the rest of the proof we will assume that $r_0 \neq 0$. It is enough to show that $r_0 \geq r_1 \geq \cdots \geq r_k$ since an analogous argument will show that $r_0 \geq s_1 \geq \cdots \geq s_l$. Suppose that $r_0 \geq r_1 \geq r_2 \geq \cdots \geq r_q$ and $r_q < r_{q+1}$ for some $0 \leq q < k$. Then, we need to show that $\bar{X}(\mu, \nu)$ is not unitary. Assume first that $r_q \neq 0$. Then,

after extracting r_q decreasing sequences $\nu_1, \nu_2, \ldots, \nu_{r_q}$ from ν^1, we can still extract a decreasing sequence ν_{r_q+1} which contains the coordinate $(4q+5)/2$ but does not contain the coordinate $(4q+1)/2$. Then, if ν_{r_q+1} has either positive coordinates smaller than $(4q+1)/2$ or some negative coordinate, we have $\dim V_{r_q+1} > 1$ and by the above argument we are done. If, on the other hand, $\dim V_{r_q+1} = 1$ the parameter ν_{r_q+1} is as in (7) with $m - j = q + 1 \geq 1$; hence, the above argument shows that $\bar{X}(\mu, \nu)$ is not unitary. If $r_q = 0$, let $s = \max\{j : r_j \neq 0 \ \text{ and } \ 0 \leq j \leq q\}$ and apply the previous argument with r_s instead of r_q. This completes the proof of the theorem. $\qquad\square$

3. Unitarity of a distinguished class of modules

In this section we restrict our attention to irreducible hermitian $(\mathfrak{g}, K)$-modules $\bar{X}(\lambda_1, \lambda_2)$ of $G = Spin(2n, \mathbb{C})$ such that $\mu = \lambda_1 - \lambda_2 = (\frac{1}{2}, \ldots, \frac{1}{2})$ and the integral system $\Delta(\lambda_1) = \Delta(\lambda_2)$ is of type $D_n \times D_n$. As we indicated in (2), this assumption implies that $\lambda_1 = (\lambda_1^L, \lambda_1^R)$, where

$$
(8) \qquad
\begin{aligned}
\lambda_1^L &= \left(k + \tfrac{1}{2}, \ldots, k + \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \ldots, -l + \tfrac{1}{2}, \ldots, -l + \tfrac{1}{2}\right), \\
\lambda_1^R &= (l, \ldots, l, \ldots, 1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1, \ldots, -k, \ldots, -k)
\end{aligned}
$$

and $\lambda_2 = (\lambda_2^L, \lambda_2^R) = (-w_0\lambda_1^R, -w_0\lambda_1^L)$, where w_0 is the longest element in $W(D_n)$, the Weyl group of D_n. Here, for any $j = 0, 1, \ldots, k$ the coordinate $j + \frac{1}{2}$ of λ_1^L as well as the coordinate $-j$ of λ_1^R occur r_j times. Similarly, for $j = 1, \ldots, l$ the coordinate $-j + \frac{1}{2}$ of λ_1^L and the coordinate j of λ_1^R occur s_j times. Recall that the numbers r_j and s_j were defined in the previous section. Also, in view of Theorem 2.1, we assume that the inequalities $r_0 \geq r_1 \geq \ldots \geq r_k$ and $r_0 \geq s_1 \geq \ldots \geq s_l$ hold.

Among these representations we consider the subclass formed by those which satisfy

$$
(9) \qquad
\begin{aligned}
r_0 = s_1 = \cdots = s_{q_r} > s_{q_r+1} = \cdots = s_{q_{r-1}} > \cdots > s_{q_2+1} = \cdots = s_{q_1}, \\
r_0 = r_1 = \cdots = r_{p_r} > r_{p_r+1} = \cdots = r_{p_{r-1}} > \cdots > r_{p_2+1} = \cdots = r_{p_1}
\end{aligned}
$$

where $r = r_0$, $s_{q_1} = r_{p_1} = 1$ and

$$
0 \leq q_r \leq p_r < q_{r-1} \leq p_{r-1} < \cdots < q_2 \leq p_2 < q_1 \leq p_1 \, .
$$

Our goal in this section is to show that these representations are unitary. The main property is that such a parameter satisfies the conclusion of Lemma 3.1.

If $\bar{X} = \bar{X}(\lambda_1, \lambda_2)$ is an irreducible representation of $Spin(2n, \mathbb{C})$ that satisfies (9) for the sequences $(p_1, p_2, \ldots, p_r)$ and $(q_1, q_2, \ldots, q_r)$, we will say that $\bar{X}$ is associated or corresponds to these sequences. Also, since the integral system $\Delta(\lambda_1)$ is of type $D_n \times D_n$ we may restrict our attention to the irreducible representations $\bar{X}(\lambda_1^L, \lambda_2^L)$ and $\bar{X}(\lambda_1^R, \lambda_2^R)$ of $Spin(n, \mathbb{C})$. Moreover, since $\bar{X}(\lambda_1^R, \lambda_2^R)$ is the hermitian dual of $\bar{X}(\lambda_1^L, \lambda_2^L)$, it is enough to consider just one of these representations.

Let $\tilde{\lambda}_1 \in W.\lambda_1^L$ and $\tilde{\lambda}_2 \in W.\lambda_2^L$ be dominant. As shown in Section 4.4 of [B] we can attach to $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ nilpotent orbits $\mathcal{O}(\tilde{\lambda}_1)$ and $\mathcal{O}(\tilde{\lambda}_2)$ in the Lie algebra of $Spin(n, \mathbb{C})$. Now, for $i = 1, 2$ let $W_{\tilde{\lambda}_i}$ denote the centralizer of $\tilde{\lambda}_i$ in W and let $w(\tilde{\lambda}_i)$ be the longest element in $W_{\tilde{\lambda}_i}$. It follows from Theorem 3.20 of [BV1] that the left cell $V^L(w(\tilde{\lambda}_i)w_0)$ contains a unique special representation σ_i of W, which occurs with multiplicity one. Then, Theorem 3.20 of [BV1] and Proposition 4.4 of [B] imply that the nilpotent orbit $\mathcal{O}(\tilde{\lambda}_i)$ is the one that corresponds to σ_i via the Springer Correspondence. We describe the nilpotent orbits in the Lie algebra of $Spin(n, \mathbb{C})$

via Young diagrams. If the Young diagram corresponding to a nilpotent orbit $\mathcal{O}$ has columns of length $x_1 \geq x_2 \geq \cdots \geq x_s$, we will write $\mathcal{O} = (x_1, x_2, \cdots, x_s)$.

Lemma 3.1. $\mathcal{O}(\tilde{\lambda}_1) = \mathcal{O}(\tilde{\lambda}_2) = (2p_1 + 2, 2q_1, \ldots, 2p_r + 2, 2q_r)$.

Proof. This result follows from an application of the algorithm given in Section 6.3 of [B]. This algorithm allow us to compute the symbol of the orbits $\mathcal{O}(\tilde{\lambda}_1)$ and $\mathcal{O}(\tilde{\lambda}_2)$ (in the sense of [L1]) from the parameters $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. Since λ_1^L and λ_2^L satisfy condition (9), this calculation gives the same symbol for both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, hence $\mathcal{O}(\tilde{\lambda}_1) = \mathcal{O}(\tilde{\lambda}_2)$. In fact, condition (9) is equivalent to the nilpotent orbits being equal. The Young diagram of the orbit can be computed from its symbol (see [L1]). □

It is known, by [BV2] and [H], how to attach to any admissible representation π of $Spin(n, \mathbb{C})$ a set in the nilpotent cone of its Lie algebra, denoted by $WF(\pi)$ and called the *wavefront set* of π. Moreover, if π is irreducible, $WF(\pi)$ is the closure of one nilpotent orbit. Then, if $\mathcal{O}$ denotes the nilpotent orbit $\mathcal{O}(\tilde{\lambda}_1) = \mathcal{O}(\tilde{\lambda}_2)$, we have the following result.

Proposition 3.1. $WF\left(\bar{X}(\lambda_1^L, \lambda_2^L)\right) = \bar{\mathcal{O}}$, *the closure of* $\mathcal{O}$.

Proof. It is enough to show that $WF\left(\bar{X}(\lambda_1^L, \lambda_2^L)\right) \subseteq \bar{\mathcal{O}}$. From this the result will follow since in view of Corollary 5.19 of [BV1] there isn't any representation with wavefront set strictly contained in $\mathcal{O}$ (at this infinitesimal character). To prove $WF\left(\bar{X}(\lambda_1^L, \lambda_2^L)\right) \subseteq \bar{\mathcal{O}}$ we proceed by induction on r, that is, the number of times that the coordinate $1/2$ occurs in λ_1^L. If $r = 1$, the $\mathfrak{a}$-parameter of $\bar{X}(\lambda_1^L, \lambda_2^L)$ has the form $\nu = \left(2p + \frac{1}{2}, 2(p-1) + \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, -2(q-1) + \frac{1}{2}, -2q + \frac{1}{2}\right)$ with $0 \leq q \leq p$. Then, if $n = p + q + 1$ and $M = GL(n, \mathbb{C})$, we know that $\bar{X}(\mu, \nu)$ is a subquotient of

$$\text{Ind}_{GL(n)}^{D_n} \left[\bar{X}_M(\mu, \nu) \otimes 1\right].$$

Hence, since $\bar{X}_M(\mu, \nu)$ is a character of $GL(n, \mathbb{C})$, we have

$$(10) \qquad WF\left(\bar{X}(\mu, \nu)\right) \subseteq \overline{\text{Ind}_{GL(n)}^{D_n}(\{0\})} = \bar{\mathcal{O}}_1$$

where $\mathcal{O}_1 = (p + q + 1, p + q + 1)$ or $(p + q + 2, p + q)$.

Now we break up the parameter ν as follows, $\nu_1 = \left(2p + \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $\nu_2 = \left(-2 + \frac{1}{2}, \ldots, -2q + \frac{1}{2}\right)$, and regard $\bar{X}(\mu, \nu)$ as a subquotient of the representation

$$\text{Ind}_{D_{p+1} \times GL(q)}^{D_n} \left[\bar{X}(\mu_1, \nu_1) \otimes \bar{X}(\mu_2, \nu_2)\right],$$

where μ_1 and μ_2 are the corresponding restrictions of $\mu = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Since $\bar{X}(\mu_1, \nu_1)$ is a finite dimensional representation of $Spin(p + 1, \mathbb{C})$ and $\bar{X}(\mu_2, \nu_2)$ is a character of $GL(q, \mathbb{C})$, we have

$$(11) \qquad WF\left(\bar{X}(\mu, \nu)\right) \subseteq \overline{\text{Ind}_{D_{p+1} \times GL(q)}^{D_n}(\mathcal{O}' \times \{0\})} = \bar{\mathcal{O}}_2$$

where $\mathcal{O}' = (2p + 2)$ and $\mathcal{O}_2 = (2p + 2, q, q)$. Now, from (10) and (11) it follows that $WF(\bar{X}(\mu, \nu)) \subseteq \bar{\mathcal{O}}$ where $\mathcal{O} = (2p + 2, 2q)$, as we wanted to prove.

Assume now that the result holds for any irreducible $(\mathfrak{g}, K)$-module satisfying condition (9) with $r_0 = r - 1$, and let $\bar{X} = \bar{X}(\lambda_1^L, \lambda_2^L)$ be associated to the sequences $(p_1, p_2, \ldots, p_r)$ and $(q_1, q_2, \ldots, q_r)$. If ν is the $\mathfrak{a}$-parameter of $\bar{X}$, we extract one coordinate from each entry of ν. This gives a decreasing sequence $\nu_1 = (2p_1 +$

$\frac{1}{2}, 2(p_1 - 1) + \frac{1}{2}, \ldots, -2(q_1 - 1) + \frac{1}{2}, -2q_1 + \frac{1}{2})$ of length $s = p_1 + q_1 + 1$. Then, if ν_2 denotes the remainder of ν and $m = n - s$, we regard $\bar{X}(\mu, \nu)$ as a subquotient of

$$\mathrm{Ind}_{GL(s) \times D_m}^{D_n} \left[\bar{X}(\mu_1, \nu_1) \otimes \bar{X}(\mu_2, \nu_2) \right],$$

where μ_1 and μ_2 are the corresponding restrictions of $\mu = (\frac{1}{2}, \ldots, \frac{1}{2})$. Now, since $\bar{X}(\mu_2, \nu_2)$ satisfies condition (9) for the sequences $(p_2, \ldots, p_r)$ and $(q_2, \ldots, q_r)$, the inductive hypothesis implies that $WF(\bar{X}(\mu_2, \nu_2)) = \bar{\mathcal{O}}_1'$ where $\mathcal{O}_1' = (2p_2 + 2, 2q_2, \ldots, 2p_r + 2, 2q_r)$. Hence,

$$(12) \qquad WF\left(\bar{X}(\mu, \nu)\right) \subseteq \overline{\mathrm{Ind}_{GL(s) \times D_m}^{D_n} \left[\mathcal{O}_1' \times \{0\}\right]} = \bar{\mathcal{O}}_1''$$

where $\mathcal{O}_1'' = (p_1 + q_1 + 1, p_1 + q_1 + 1, 2p_2 + 2, \ldots, 2q_r)$ or $(p_1 + q_1 + 2, p_1 + q_1, 2p_2 + 2, \ldots, 2q_r)$.

We can also regard $\bar{X}(\mu, \nu)$ as a subquotient of

$$\mathrm{Ind}_{GL(s') \times D_{m'}}^{D_n} \left[\bar{X}(\mu_1', \nu_1') \otimes \bar{X}(\mu_2', \nu_2') \right],$$

where $\nu_1' = \left(2p_2 + \frac{1}{2}, 2(p_2 - 1) + \frac{1}{2}, \ldots, -2(q_1 - 1) + \frac{1}{2}, -2q_1 + \frac{1}{2}\right)$ is a decreasing sequence of length $s' = p_2 + q_1 + 1$, $m' = n - s'$, ν_2' is the remainder of ν after extracting ν_1' and μ_i' $(i = 1, 2)$ are the corresponding restrictions of μ. Now, since $\bar{X}(\mu_2', \nu_2')$ satisfies condition (9) for the sequences $(p_1, p_3, \ldots, p_r)$ and $(q_2, q_3, \ldots, q_r)$, the inductive hypothesis implies that $WF\left(\bar{X}(\mu_2', \nu_2')\right) = \bar{\mathcal{O}}_2'$ where $\mathcal{O}_2' = (2p_1 + 2, 2q_2, \ldots, 2p_r + 2, 2q_r)$. Hence, since $\bar{X}(\mu_1', \nu_1')$ is a character of $GL(s', \mathbb{C})$, we have

$$(13) \qquad WF\left(\bar{X}(\mu, \nu)\right) \subseteq \overline{\mathrm{Ind}_{GL(s') \times D_{m'}}^{D_n} \left[\mathcal{O}_2' \times \{0\}\right]} = \bar{\mathcal{O}}_2'',$$

where $\mathcal{O}_2'' = (2p_1 + 2, p_2 + q_1 + 1, p_2 + q_1 + 1, 2q_2, \ldots, 2p_r + 2, 2q_r)$. From (12) and (13) it follows that $WF\left(\bar{X}(\mu, \nu)\right) \subseteq \bar{\mathcal{O}}$, since any nilpotent orbit contained in $\bar{\mathcal{O}}_1''$ and $\bar{\mathcal{O}}_2''$ must be contained in $\bar{\mathcal{O}}$. This completes the induction and the proof of the proposition. □

We now consider the set $\mathfrak{X}(\lambda_1^L, \lambda_2^L)$ of all $(\mathfrak{g}, K)$-modules with infinitesimal character $(\lambda_1^L, \lambda_2^L)$ and wavefront set contained in $\bar{\mathcal{O}}$. That is,

$$(14) \qquad \mathfrak{X}(\lambda_1^L, \lambda_2^L) = \left\{ \bar{X}(\lambda_1^L, w\lambda_2^L) \; : \; w \in W \text{ and } WF\left(\bar{X}(\lambda_1^L, w\lambda_2^L)\right) \subseteq \bar{\mathcal{O}} \right\}.$$

Then, if $V^L(w(\tilde{\lambda}_1)w_0)$ and $V^R(w(\tilde{\lambda}_2)w_0)$ are respectively the left and right cell representations of W associated to $w(\tilde{\lambda}_1)w_0$ and $w(\tilde{\lambda}_2)w_0$, we have

Proposition 3.2. $\mid \mathfrak{X}(\lambda_1^L, \lambda_2^L) \mid = \dim \mathrm{Hom}_W \left[V^L(w(\tilde{\lambda}_1)w_0) : V^R(w(\tilde{\lambda}_2)w_0) \right] = 1.$

Proof. The first equality is a result of Lusztig (see [L2] and Proposition 5.25 of [BV1]). To prove the second equality we recall that it follows from Section 6.3 of [B] that the symbol of the left cell $V^L(w(\tilde{\lambda}_1)w_0)$ is

$$(15) \qquad (a_0, a_1)(a_2, a_3)(a_4, a_5) \cdots (a_{2n}, a_{2n+1}) \qquad (a_i \leq a_{i+1})$$

with $a_{2i-1} < a_{2i}$ for $1 \leq i < n$, and the symbol of $V^L(w(\tilde{\lambda}_2)w_0)$, which is isomorphic to the right cell $V^R(w(\tilde{\lambda}_2)w_0)$, is

$$(16) \qquad (a_0, a_{2n+1})(a_1, a_2)(a_3, a_4) \cdots (a_{2n-1}, a_{2n})$$

with $a_{2i} < a_{2i+1}$ for $1 \leq i \leq n-1$. Then, the left cells $V^L(w(\tilde{\lambda}_i)w_0)$ (for $i = 1, 2$) are the sum of all the irreducible representations of W with symbol

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \end{pmatrix}$$

satisfying $\{\lambda_1, \ldots, \lambda_{n+1}, \mu_1, \cdots, \mu_{n+1}\} = \{a_0, a_1, \ldots, a_{2n}, a_{2n+1}\}$ and such that $\mu_1, \ldots, \mu_{n+1}$ contains one number from each of the $n+1$ pairs which form the symbol of the cell (see [L2]). From this it follows that the only representation that occurs in both left cells is the one with symbol

$$\begin{pmatrix} a_1 & a_3 & \cdots & a_{2n+1} \\ a_0 & a_2 & \cdots & a_{2n} \end{pmatrix}.$$

This completes the proof of the proposition. $\qquad\square$

Theorem 3.1. *Let $\bar{X} = \bar{X}(\lambda_1, \lambda_2)$ be an irreducible representation of D_{2n} that satisfies condition (9) for the sequences $(p_1, p_2, \ldots, p_r)$ and $(q_1, q_2, \ldots, q_r)$, and suppose that $p_1 = q_1$. Then, if $\bar{Y}$ denotes the irreducible representation of D_{2m} corresponding to the sequences $(p_2, p_3, \ldots, p_r)$ and $(q_2, q_3, \ldots, q_r)$ (here $m < n$), and σ_{2s} denotes the Stein complementary series of $GL(2s, \mathbb{C})$ (here $s = 2p_1 + 1$) with $\mathfrak{a}$-parameter $\nu = (\nu_1, -\nu_1)$ and $\nu_1 = (2p_1 + \frac{1}{2}, \cdots, -2p_1 + \frac{1}{2})$ (see [V1] for the definition), we have*

$$(17) \qquad\qquad \bar{X} \cong \mathrm{Ind}_{D_{2m} \times GL(2s)}^{D_{2n}} \left[\bar{Y} \otimes \sigma_{2s} \right].$$

In particular, it follows that $\bar{X}$ is unitary if and only if $\bar{Y}$ is unitary.

Proof. It is enough to show that the induced representation on the right hand side of (17) is irreducible. Let (λ_1', λ_2') be the parameter of $\bar{Y}$. Since the integral system $\Delta(\lambda_1')$ is of type $D_m \times D_m$ we consider, as before, the irreducible representations $\bar{Y}^L = \bar{X}((\lambda_1')^L, (\lambda_2')^L)$ and $\bar{Y}^R = \bar{X}((\lambda_1')^R, (\lambda_2')^R)$ of $Spin(m, \mathbb{C})$. Similarly, we let σ_{2s}^L (respectively σ_{2s}^R) denote the irreducible representation of $GL(s, \mathbb{C})$ with $\mathfrak{a}$-parameter ν_1 (respectively $-\nu_1$).

To show that the representation on the right hand side of (17) is irreducible, it is enough to show that

$$I^L = \mathrm{Ind}_{D_m \times GL(s)}^{D_n} \left[\bar{Y}^L \otimes \sigma_{2s}^L \right]$$

is irreducible. Now, in view of Proposition 3.1, $WF(\bar{Y}^L) = \bar{\mathcal{O}}_1$ where $\mathcal{O}_1 = (2p_2 + 2, 2q_2, \ldots, 2p_r + 2, 2q_r)$ and since σ_{2s}^L is a character on $GL(s, \mathbb{C})$,

$$WF(I^L) \subseteq \overline{\mathrm{Ind}_{D_m \times GL(s)}^{D_n} [\mathcal{O}_1 \times \{0\}]} = \bar{\mathcal{O}},$$

where $\mathcal{O} = (2p_1 + 2, 2q_1, 2p_2 + 2, 2q_2, \ldots, 2p_r + 2, 2q_r)$; that is, $WF(I^L) = WF\left(\bar{X}(\lambda_1^L, \lambda_2^L)\right)$. Hence, since the infinitesimal character of I^L is $(\lambda_1^L, \lambda_2^L)$, it follows from Proposition 3.2 that I^L is irreducible. This completes the proof of the proposition. $\qquad\square$

Theorem 3.2. *Let $\bar{X} = \bar{X}(\lambda_1, \lambda_2)$ be an irreducible representation of D_{2n} satisfying condition (9) for the sequences $(p_1, p_2, \ldots, p_r)$ and $(q_1, q_2, \ldots, q_r)$, and suppose that $p_1 > q_1$. Let $\bar{Y}$ be the irreducible representation of $D_{2(n-1)}$ corresponding to the sequences $(p_1 - 1, p_2, \ldots, p_r)$ and $(q_1, q_2, \ldots, q_r)$, let σ_{2s} be the Stein complementary series representation of $GL(2s, \mathbb{C})$ (here $s = 2p_1$) with $\mathfrak{a}$-parameter $\nu = (\nu_1, -\nu_1)$*

and $\nu_1 = (2(p_1 - 1) + \frac{1}{2}, \cdots, -2p_1 + \frac{1}{2})$ *and let* σ_{2s+2} *be the Stein complementary series representation of* $GL(2s + 2, \mathbb{C})$ *with* $\mathfrak{a}$-*parameter* $\nu' = (\nu'_1, -\nu'_1)$ *where* $\nu'_1 = (2p_1 + \frac{1}{2}, \ldots, -2p_1 + \frac{1}{2})$. *Then,*

$$(18) \qquad \mathrm{Ind}_{D_{2n} \times GL(2s)}^{D_{2(n+s)}} \left[\bar{X} \otimes \sigma_{2s} \right] \cong \mathrm{Ind}_{D_{2(n-1)} \times GL(2s+2)}^{D_{2(n+s)}} \left[\bar{Y} \otimes \sigma_{2s+2} \right]$$

and it is irreducible. In particular, $\bar{X}$ *is unitary if and only if* $\bar{Y}$ *is unitary.*

Proof. Since the integral system $\Delta(\lambda_1)$ is of type $D_n \times D_n$, as in the previous proposition, we consider the representations $\bar{X}^L = \bar{X}(\lambda_1^L, \lambda_2^L)$ and $\bar{X}^R = \bar{X}(\lambda_1^R, \lambda_2^R)$ of D_n and let σ_{2s}^L (respectively σ_{2s}^R) denote the irreducible representations of $GL(s, \mathbb{C})$ with $\mathfrak{a}$-parameter ν_1 (respectively $-\nu_1$). Then, to prove that the representation on the left hand side of (18) is irreducible, it is enough to show that

$$I^L = \mathrm{Ind}_{D_n \times GL(s)}^{D_{n+s}} \left[\bar{X}^L \otimes \sigma_{2s}^L \right]$$

is irreducible. Now, in view of Proposition 3.1 we have $WF(\bar{X}^L) = \bar{\mathcal{O}}_1$ where $\mathcal{O}_1 = (2p_1 + 2, 2q_1, \ldots, 2p_r + 2, 2q_r)$. Then, since σ_{2s}^L is a character on $GL(s, \mathbb{C})$, we have

$$WF(I^L) \subseteq \overline{\mathrm{Ind}_{D_n \times GL(s)}^{D_{n+s}} [\mathcal{O}_1 \times \{0\}]} = \bar{\mathcal{O}},$$

where $\mathcal{O} = (2p_1 + 2, 2p_1, 2p_1, 2q_1, 2p_2 + 2, \ldots, 2q_r)$. This nilpotent orbit is obtained by adding 2 to the largest $s = 2p_1$ rows of the Young diagram of $\mathcal{O}_1$. On the other hand, if (λ'_1, λ'_2) is the parameter on $D_{2(n+s)}$ of the form (9) corresponding to the sequences $(p_1, p_1 - 1, p_2, \ldots, p_r)$ and $(p_1, q_1, q_2, \ldots, q_r)$, the infinitesimal character of I^L is $((\lambda'_1)^L, (\lambda'_2)^L)$ and $WF\left(\bar{X}((\lambda'_2)^L, (\lambda'_2)^L) \right) = \bar{\mathcal{O}}$. Then, in view of Proposition 3.2, I^L is irreducible and therefore,

$$\bar{X}(\lambda'_1, \lambda'_2) \cong \mathrm{Ind}_{D_{2n} \times GL(2s)}^{D_{2(n+s)}} \left[\bar{X} \otimes \sigma_{2s} \right] .$$

Now, since the sequences $(p_1, p_1 - 1, p_2, \ldots, p_r)$ and $(p_1, q_1, q_2, \ldots, q_r)$ corresponding to the parameter (λ'_1, λ'_2) satisfy the hypothesis of Theorem 3.1, a direct application of this theorem gives the equality in (18). This completes the proof of the theorem. $\qquad \square$

We are now in a position to prove the main result of this section.

Theorem 3.3. *Let* $\bar{X} = \bar{X}(\lambda_1, \lambda_2)$ *be an irreducible representation of* $Spin(2n, \mathbb{C})$ *satisfying condition* (9) *for the sequences* $(p_1, p_2, \ldots, p_r)$ *and* $(q_1, q_2, \ldots, q_r)$. *Then,* $\bar{X}$ *is unitary.*

Proof. We do an induction on r, the number of times that the coordinate $1/2$ occurs in λ_1^L, and the difference $p_1 - q_1$. For $r = 1$ and $p_1 = q_1$, $\bar{X}$ is unitary by Theorem 3.1; in fact, $\bar{X}$ is unitarily induced irreducible from the Stein complementary series σ_{2s} on $GL(2s, \mathbb{C})$ (with $s = 2p_1 + 1$). Suppose $p_1 > q_1$. Denote by $(p'_1, \cdots, p'_r)$ and $(q'_1, \cdots, q'_r)$ the sequences corresponding to $\bar{Y}$ defined in Theorem 3.2. Then Theorem 3.2 says that $\bar{X}$ is unitary if and only if $\bar{Y}$ is unitary. Now, since $p'_1 - q'_1 < p_1 - q_1$, the inductive hypothesis implies that $\bar{Y}$ (and therefore $\bar{X}$) is unitary.

If, on the other hand, $r > 1$ and $p_1 = q_1$, let $\bar{Y}$ be the irreducible representation defined in Theorem 3.1 with $r' = r - 1$. Then Theorem 3.1 says that $\bar{X}$ is unitary if and only if $\bar{Y}$ is unitary. Now, since $r' < r$, the inductive hypothesis implies that

$\bar{Y}$ (and therefore $\bar{X}$) is unitary. This completes the induction step and the proof of the theorem. $\qquad \blacksquare$

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